International Journal of Theoretical Physics, Vol. 29, No. 12, 1990

Newtonian Motion and Lorentz–Dirac Equation

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Received March 5, 1990

Newtonian relations between force and acceleration are confronted with the third-order Lorentz–Dirac equation of motion of an elementary classical point charge. An example of such a charge that accelerates in a region where external fields vanish is discussed.

1. INTRODUCTION

The equation of motion of an elementary classical point charge is a debated subject. The third-order Lorentz–Dirac (LD) equation is recognized by most people working in the field as the required equation (see, e.g., Dirac, 1938; Landau and Lifshitz, 1962; Rohrlich, 1965; Pearle, 1982). This equation has been criticized for its runaway solutions and for its third-order non-Newtonian form. Several authors (Eliezer, 1948; Mo and Papas, 1971; Bonnor, 1974; Herrera, 1977) have suggested alternative second-order differential equations as candidates for equations of motion of an elementary classical point charge.

It has recently been proved by Comay (1990*a*) that in the case of a scattering process of a charge moving rectilinearly in an external electrostatic field, runaway solutions are incompatible with the LD equation. On the other hand, the four second-order Newtonian equations mentioned above as substitutes for the LD equation have already been proved by Huschilt and Baylis (1974) and by Comay (1987, 1990*b*) to be unphysical. These results support the acceptance of the LD equation as *the* law of motion of elementary classical point charges.

In the present work the LD equation is taken as the correct law of motion of an elementary classical point charge. On this basis, it is proved that consequences of this third-order equation cannot be reconciled with

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0020-7748/90/1200-1427$06.00/0

Newtonian notions of linked relations between acceleration and *external* forces. Hereafter, the requirement saying that an elementary classical point charge must move inertially in a region where external fields vanish is called the Newtonian requirement.

The following discussion applies units where the speed of light takes the value $c = 1$. Greek indices range from 0 to 3 and Latin ones range from 1 to 3. The metric is diagonal and its entries are $(1, -1, -1, -1)$. The symbol $_{,\nu}$ denotes the partial differentiation with respect to x^ν and an upper dot denotes the differentiation with respect to the particle's invariant time τ. The symbols v^μ and a^μ denote 4-velocity and 4-acceleration of a particle, respectively.

The structure of the discussion is as follows. Some properties of the LD equation are pointed out in Section 2. An experiment where an elementary classical point charge accelerates in an external field-free region is discussed in Section 3. The contents of Section 4 substantiate the results of the previous one. In Section 5 it is explained why classical electrodynamics of an elementary point charge cannot generally satisfy the Newtonian requirement. Concluding remarks are included in the last section.

2. SOME PROPERTIES OF THE LD EQUATION

The covariant form of the LD equation of a charge q is

$$\tfrac{2}{3}q^2\dot{a}^\mu = Ma^\mu - qF^{\mu\nu}_{\text{ext}}v_\nu - \tfrac{2}{3}q^2(a^\alpha a_\alpha)v^\mu \tag{1}$$

where M denotes the mass of q and $F^{\mu\nu}_{\text{ext}}$ is the tensor of electromagnetic fields associated with external sources. This is a third-order differential equation of the charge coordinate x^μ with respect to its invariant time τ. evidently, using the relation

$$\frac{d}{d\tau} = \gamma\frac{d}{dt} \tag{2}$$

where $\gamma = (1 - v^2)^{-1/2}$, one can cast it into a third-order equation with respect to the laboratory time t. The LD equation is used here in the case of a single charge moving rectilinearly in an external electrostatic field. As proved by Comay (1990*a*), a scattering process of this kind, where a charge moves inertially at $\tau = -\infty$, before it approaches the interaction region, ends in asymptotic inertial motion at $\tau \to \infty$, when the interaction stops and the charge moves toward infinity. This physically agreeable property supports the acceptance of the LD equation as an equation describing correctly the motion of classical point charges.

The asymptotic inertial motion of a charge obeying the LD equation is used here in a proof of its compatibility with energy conservation. To this end, the 0-component of the LD equation (1) is integrated on the interval $(-\infty, \infty)$.

The term on the left-hand side of (1) yields

$$\int_{-\infty}^{\infty} \frac{da^0}{d\tau}\, d\tau = a^0(\infty) - a^0(-\infty) = 0 \tag{3}$$

where the null result is derived from the vanishing acceleration at $x = \pm\infty$. This conclusion follows the initial value at the remote past and the asymptotic inertial motion derived by Comay (1990*a*).

As is well known, the 4-velocity takes the form

$$v^\mu = \gamma(1, v_x, v_y, v_z) \tag{4}$$

Using this relation for the first term on the right-hand side of (1), one readily finds

$$\int_{-\infty}^{\infty} Ma^0\, d\tau = M\gamma(\infty) - M\gamma(-\infty) \tag{5}$$

Result (5) is the decrease of the charge's self-energy $M\gamma$ during the entire motion.

For the next quantity, one finds

$$\int_{-\infty}^{\infty} F^{0i} v_i\, d\tau = \int_{-\infty}^{\infty} \mathbf{E} \cdot \frac{d\mathbf{x}}{d\tau}\, d\tau = \int_{-\infty}^{\infty} -\text{grad}\,\Phi \cdot d\mathbf{x} = \Phi(-\infty) - \Phi(\infty) = 0 \tag{6}$$

where Φ denotes the electrostatic potential associated with external charges. This potential vanishes at infinity and the null value of (6) follows.

The last quantity is

$$-\int_{-\infty}^{\infty} \tfrac{2}{3}q^2(a^\alpha a_\alpha)v^0\, d\tau = \int_{-\infty}^{\infty} \frac{dP^0_{\text{rad}}}{d\tau}\, d\tau = P^0_{\text{rad}} \tag{7}$$

where P^0_{rad} denotes the energy radiated from a system where just a single charge accelerates [see, e.g., Landau and Lifshitz (1962, p. 222) or Rohrlich (1965, p. 111), but notice the different metric used therein].

The LD equation (1) and the results (3) and (5)–(7) prove that the decrease in the charge's mechanical energy during the entire process equals the energy radiated by it. Hence, energy conservation is established for the entire scattering process discussed here. This conclusion shows an important physical property of the LD equation: in the case of a charge moving rectilinearly in an electrostatic external field, a solution that ends with an inertial motion at infinity yields (3) and energy conservation follows. This satisfactory property of the LD equation is found useful in a later part of the discussion.

3. AN EXPERIMENT

The inertial motion of a charge obeying the LD equation is derived by Comay (1990*a*) for asymptotic regions where external electrostatic fields become negligible. At these regions these fields decrease like $1/r^n$, where n is an integer greater then 1. This is a crucial element of the derivation of inertial motion in asymptotic regions. It turns out that the derivation does not apply to the motion of a charge at field-free portions of space which are located near the interaction region. The following experiment tests a case like this.

Consider an elementary classical point charge q that moves along the x axis from $x=-\infty$ toward $x=\infty$. A motionless sphere whose radius is R is made of an insulating material and its center coincides with the origin. The sphere is covered uniformly with electric charge Q. The path along which q moves is split into three intervals, $(-\infty, -R]$, $(-R, R)$, and $[R, \infty)$. The external field at these intervals is $-Q/x^2$, 0, and Q/x^2, respectively. Our purpose is to find the motion of q at the field-free interval $(-R, R)$.

Evidently, the acceleration a of q does not vanish identically at points like $x=-R$, where it enters the field-free region. This property can be proved in the following way. Assume $a=0$ at $x=-R$. Consider an analogous experiment where the charged sphere is replaced by a similar sphere whose radius is

$$R' = R - \delta \tag{8}$$

and the two spheres are covered with the same amount of charge. Evidently, along the interval $(-\infty, -R]$, the LD equation takes the same form in the two cases and the motion of q is the same. If the acceleration of q does not vanish at R', then the required result holds. If, on the other hand, for every value $\delta \in [0, \varepsilon]$ used in (8), $a=0$ at the corresponding R', then one finds that q moves inertially along the interval $[-R, -R+\varepsilon]$, where the external field is nonzero. It is shown that this result is incompatible with the LD equation (1).

If the 3-acceleration vanishes, then γ is a constant and an inspection of (4) shows that the 4-acceleration vanishes, too. As assumed above, the acceleration vanishes at *an interval* $[0, \varepsilon]$. This property means that the 4-acceleration as well as its derivative vanish there. Substituting $a^\mu = \dot{a}^\mu = 0$ into the LD equation (1), one finds a single nonzero term $qE\gamma v$. Hence, an inertial motion along an interval where an external electric field that is parallel to the motion does not vanish is inconsistent with the LD equation.

The foregoing discussion proves that, at $x=-R$, one generally finds that the acceleration $a \neq 0$. In the given problem, the third-order LD equation is piecewise continuously differentiable at the interval $(-\infty, \infty)$. Let τ_1 be the invariant time of q when it passes $x=-R$. It is shown in the next section that the LD equation has a unique solution at a small time interval which includes τ_1. Therefore, an integration shows that at τ_1, the coordinate, velocity, and acceleration of q vary continuously.

Using this outcome, let us turn to the motion of q along the interval $(-R, R)$ where the external electrostatic field vanishes. The initial values at τ_1 are $x=-R$, $v>0$, and $a \neq 0$. Assume that q reaches an inner point $x=-R+\eta$ at $\tau'=\tau_1+\theta$. It follows that, at this point, the 4-acceleration is

$$a^\mu(\tau') = a^\mu(\tau_1) + \int_{\tau_1}^{\tau'} \frac{da^\mu}{d\tau}\, d\tau \tag{9}$$

The derivative $da^\mu/d\tau$ can be obtained from the LD equation (1). Obviously, since $a^\mu(\tau_1) \neq 0$ and $da^\mu/d\tau$ is bounded at $(-R, -R+\eta)$, one can always define a small enough value for η so that the acceleration does not vanish at this interval. This result means that an experiment can be constructed so that *a charge obeying the LD equation accelerates in a region where external fields vanish.*

4. A FURTHER DISCUSSION OF THE ACCELERATION

The following arguments indicate that the last conclusion is an inherent property of the LD equation. In particular, it is explained here why one should not try compelling the charge to obey the Newtonian requirement.

One point relies on the structure of theoretical physics. Here, some basic principles yield mathematical relations like laws of motion. Having these results, one abides by mathematical rules while solving specific cases where the laws of motion are tested. This approach is applied to the specific example described in the previous section.

In the case of a charge moving rectilinearly from $x=-\infty$ toward $x=\infty$ in an external electrostatic field, one can eliminate the time from the LD equation (1) and cast it into a second-order equation where the coordinate x is the independent variable

$$\tfrac{2}{3}q^2(\gamma^2-1)^{1/2}\gamma'' = M\gamma' - qE(x) \tag{10}$$

and γ is defined after (2) [see Section 2 of Comay (1990*a*)]. At $x=-R$, the velocity as well as γ^2-1 are greater than zero and equation (10) satisfies the Lipschitz condition at a small region which includes this point. Therefore, the proof of existence and uniqueness of solutions applies. In particular, the proof holds for the experiment discussed in the previous section, where the differential equation is piecewise continuous [see, e.g., Coddington and Levinson, 1955).

Having a unique solution to the LD equation (10) means that (1) has such a solution, too. Hence, one can use the bounded values of the right-hand side of (1) and find that

$$a^{\mu} = \int \frac{da^{\mu}}{d\tau}\, d\tau$$

is continuous and (9) holds. It follows that the conclusion obtained in the previous section stating that a charge can accelerate in a region where external fields vanish has a sound mathematical basis.

The next point shows that if the LD equation is valid, then the Newtonian requirement is unphysical. To this end, assume that the charge obeys the LD equation at all regions, except along a small interval $[-R-\rho, -R+\rho]$. At this interval the charge "surrenders" to the Newtonian requirement and exits at $x=-R+\rho$ with a null acceleration. This assumption means that along this short interval, a second-order Newtonian equation written in terms of external fields controls the motion. The scheme is tested in the limit where $\rho\to 0$.

Let us test energy conservation in the process described above. Since $\rho\to 0$ and the equation is bounded, one can ignore the contributions made along the interval $[-R-\rho, -R+\rho]$. Thus, instead of (3), one finds

$$\begin{aligned} &\int_{-\infty}^{-R-\rho} \frac{da^0}{d\tau}\, d\tau + \int_{-R+\rho}^{\infty} \frac{da^0}{d\tau}\, d\tau \\ &\quad = a^0(-R-\rho) - a^0(-\infty) + a^0(\infty) - a^0(-R+\rho) \\ &\quad = a^0(-R-\rho) \end{aligned} \tag{11}$$

In this calculation a^0 vanishes at infinity because of initial values at $\tau=-\infty$ and due to the results obtained by Comay (1990*a*). It also vanishes at $x=-R+\rho$, in accordance with the assumed validity of the Newtonian requirement. An inspection of the final expression of (5)-(7) shows that, since the continuity of these quantities is sustained, the corresponding expressions do not change.

Comparing this analysis with that of Section 2, where energy conservation by the LD equation is proved, it is found that if, along a very short interval, the moving charge "surrenders" to the Newtonian requirement, then an extra term $\frac{2}{3}q^2a^0(-R-\rho)$ appears and energy balance is destroyed. Indeed, as shown in the previous section, at $x=-R-\rho$, a^0 as well as the extra term do not vanish identically. This conclusion means that one cannot violate the LD equation along a short interval in a way that complies with the Newtonian requirement and with energy conservation as well.

5. CONTINUOUSLY CHARGED PARTICLES AND ELEMENTARY POINT PARTICLES

The comparison of particles made of continuously distributed charged matter and elementary classical point particles provides another reason for the abandonment of the Newtonian requirement. Moreover, it shows that this conclusion emerges from fundamental properties of classical electrodynamics and should not be considered as a surprise.

Let us examine the motion of a small classical particle whose charge is distributed continuously at its volume. Hereafter, a charge like this is called a C-charge and an elementary classical point charge is called a P-charge. Classical electrodynamics of continuously distributed charged matter is a closed theory. Its equations of motion are Maxwell equations

$$F^{\mu\nu}_{,\nu} = -4\pi J^{\mu} \tag{12}$$

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0 \tag{13}$$

and the Lorentz force

$$ma^{\mu} = \int_V F^{\mu\nu} J_\nu \, d^3r = qF^{\mu\nu} v_\nu \tag{14}$$

where V denotes the particle's volume. In the final expression $F^{\mu\nu}$ represents the average fields tensor at V.

Constructing energy-momentum tensors of fields and of matter, one takes the 4-divergence of their sum and proves energy-momentum conservation

$$T^{\mu\nu}_{(f),\nu} + T^{\mu\nu}_{(m),\nu} = 0 \tag{15}$$

(Landau and Lifshitz, 1962, pp. 91–95). A property of the theory of continuously distributed charge is that one can consider an infinitesimal amount of charge dq, ignore terms depending on higher powers of this quantity, and derive linear expressions like (14).

Let us examine the motion of a C-charge q and split the field exerted on it. Thus, the Lorentz force takes the following form:

$$ma^{\mu} = \int_V F^{\mu\nu}_{\text{ext}} J_\nu \, d^3r + \int_V F^{\mu\nu}_{\text{int}} J_\nu \, d^3r \tag{16}$$

where $F^{\mu\nu}_{\text{ext}}$ and $f^{\mu\nu}_{\text{int}}$ are associated with charge located outside and inside the volume of q, respectively. Our purpose is to substitute $f^{\mu\nu}_{\text{int}}$ by quantities depending on appropriate derivatives of the coordinates of q. Expanding the retarded potentials of the charge of q as a power series in c^{-1}, Landau and Lifshitz (1962) prove that if the series derived for the fields are truncated after the quadratic terms of c^{-1}, then one obtains the Darwin Lagrangian,

$$L=\frac{1}{2}\sum_j m_j v_j^2+\frac{1}{8c^2}\sum_j m_j v_j^4-\sum_{j>l}\frac{q_j q_l}{R_{jl}} + \sum_{j>l}\frac{q_j q_l}{2c^2 R_{jl}}\left[\mathbf{v}_j\cdot\mathbf{v}_l+\frac{(\mathbf{v}_j\cdot\mathbf{R}_{jl})(\mathbf{v}_l\cdot\mathbf{R}_{jl})}{R_{jl}^2}\right] \tag{17}$$

(Landau and Lifshitz, 1962, pp. 190–193). The Darwin Lagrangian depends on the instantaneous radius vector $\mathbf{R}_{jl}$ between two volume elements ΔV_j and ΔV_l of q. Hence, the forces derived from it vanish in pairs and do not make a contribution to the motion of the center of mass of the particle.

Retaining terms up to c^{-3} in the series, Landau and Lifshitz (1962) obtains the Abraham terms of the LD equation (Rohrlich, 1965, p. 18)

$$\Gamma^\mu=\tfrac{2}{3}q^2[\dot{a}^\mu+(a^\alpha a_\alpha)v^\mu] \tag{18}$$

It is easy to see that higher terms of the expansions in c^{-1} can be ignored if the spatial size of q is small enough.

This analysis proves that, in the case of such a small C-charge, the LD equation (1), which is written in terms of $F^{\mu\nu}_{\text{ext}}$, is equivalent to the Lorentz force (14), which depends on the entire field tensor $F^{\mu\nu}$. It is further shown that the overall force exerted on a C-charge by retarded fields associated with its own charge equal (18) and *does not vanish identically.* Therefore, if a C-charge enters a region where $F^{\mu\nu}_{\text{ext}}=0$, then the Lorentz force (14) as well as its equivalent LD equation (1) prove that the particle may accelerate there, due to the nonzero contribution of $F^{\mu\nu}_{\text{int}}$. The form of the self-force (18) shows that it may provide a reason for its own existence. Indeed, it can be seen from (18) that this force does not vanish if the acceleration or its derivative are nonzero. On the other hand, if $F^{\mu\nu}_{\text{ext}}=0$ and the force (18) associated with $F^{\mu\nu}_{\text{int}}$ does not vanish, then a C-charge accelerates.

As stated in Section 2, the LD equation is accepted here as the correct equation of P-charges. It follows that the proof showing that a C-charge may accelerate at regions where $F^{\mu\nu}_{\text{ext}}=0$ holds for P-charges, too, because P-charges, like very small C-charges, satisfy the LD equation (1).

The discussion carried out in this section relies on retardation of electromagnetic fields. It proves that if terms associated with c^{-3} are not ignored in power series of fields, then *the overall self-force does not vanish* *identically*. This property of classical electrodynamics is the root of the peculiar property of charge motion where the Newtonian requirement does not hold.

6. CONCLUDING REMARKS

The discussion carried out in this work takes the following course.

It relies on the conclusions obtained by Comay (1990*b*) showing that a second-order Newtonian equation cannot depend linearly on external fields, that all suggested nonlinear equations of this kind are unphysical, and that such Newtonian equations encounter a problem that looks unsolvable: they must substitute correctly the *linear* dependence of the Lorentz force on $F^{\mu\nu}_{\text{int}}$ by a *nonlinear* dependence on $F^{\mu\nu}_{\text{ext}}$.

The third-order LD equation is *proved* by Landau and Lifshitz (1962) to be based on the right substitution. Assuming that classical electrodynamics is a self-consistent theory, the LD equation (1) is considered here as the correct law of motion of point charges. A simple experiment where a charge obeying this equation accelerates in a region where external fields vanish is described.

It is explained further why an attempt to impose inertial motion on a charge moving in a bounded region where external fields vanish violates mathematical laws as well as the law of energy conservation. The origin of the results obtained here is related to the distinction between the Lorentz force (14), which is written in terms of the *entire* field tensor $F^{\mu\nu}$, and the LD equation (1), which depends on *external* fields $F^{\mu\nu}_{\text{ext}}$. As proved by Landau and Lifshitz, the overall force exerted on a charge by retarded self-fields $F^{\mu\nu}_{\text{int}}$ *does not vanish*. It follows that a vanishing external field is not a sufficient condition for inertial motion.

As is well known, the establishment of classical electrodynamics, which culminated in the formulation of special relativity, showed that Newtonian mechanics is conceptually wrong and can be useful only as an approximation that is rather good in certain cases. The present work discusses another aspect of Newtonian mechanics. This point boils down to the idea of a null self-force. This idea is commonplace in mechanics, but, as proved by Landau and Lifshitz (1962), it does not hold in electrodynamics, where the self-force of retarded fields may be nonzero. On the other hand, this self-force does not lead to runaway solutions. As proved by Comay (1990*a*), in the case of a rectilinear motion, the LD equation satisfies *asymptotic* inertial motion as well as energy conservation by the entire process.

It is interesting to note that in quantum mechanics, the notion of force does not play a central role and the equations of motion are written in terms of potentials. Here force appears in an indirect manner in cases where the classical limit holds. In these cases, quantum mechanics yields Ehrenfest's theorem, which agrees with the Lorentz force. Schiff (1955) concludes that "a wave packet moves like a classical particle if it is sufficiently well localized so that the electromagnetic fields change by a negligible amount over its dimensions."

Similar results concerning the notion of force are obtained on a purely classical level. It is shown that acceleration is not linked to the existence of *external* force. On the other hand, as proved by Comay (1990*a*), the *asymptotic* motion of a classical charge is dominated by the part of the Lorentz force associated with external fields, whereas radiation reaction terms become negligible. Obviously, in asymptotic regions, external fields can be considered uniform over the dimensions of a wave packet. It follows that an analysis of the asymptotic motion is an example where quantum mechanics as well as classical theory show that the motion of a charge is determined by the part of the Lorentz force associated with external fields.

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